QP CODE 210011

# PILLAI COLLEGE OF ENGINEERING, NEW PANVEL
(Autonomous) (Accredited 'A+' by NAAC)
END SEMESTER EXAMINATION
SECOND HALF 2021

BRANCH: FE (COMP/IT/ECS)

Subject: Engineering Mathematics – I
Max. Marks: 60
Time: 02.00 Hours
Date: 04-04-2022

N.B 1. Q.1 is compulsory
2. Attempt any two from the remaining three questions

| | | M | BT | CO |
|---|---|---|---|---|
| Q.1. | Attempt all | | | |
| a) | If $\alpha, \alpha^2, \alpha^3, \alpha^4$ are the roots of $x^5 - 1 = 0$,find them and show that $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$. | 5 | 3 | 1 |
| b) | If $\bar{f} = x^2zi - 2y^3z^2j + xy^2zk$, find $div\bar{f}$ and $curl\bar{f}$ at (2,3,4). | 5 | 1,3 | 3 |
| c) | If $u = f(ax - by, by - cz, cz - ax)$ then find $\frac{1}{a}\frac{\partial u}{\partial x} + \frac{1}{b}\frac{\partial u}{\partial y} + \frac{1}{c}\frac{\partial u}{\partial z}$. | 5 | 2 | 4 |
| d) | Test for consistency if consistent solve $4x - 2y + 6z = 8, x + y - 3z = -1, 15x - 3y + 9z = 21$ | 5 | 3 | 5 |
| Q.2. | Attempt all | | | |
| a) | Express the following matrix A as sum of symmetric and skew symmetric matrix where $A = \begin{bmatrix} 1 & 5 & 7 \\ -1 & -2 & -4 \\ 8 & 2 & 13 \end{bmatrix}$ | 4 | 3,5 | 5 |
| b) | Solve the following transcendental equation by Newton Raphson Method $\cos(x) - xe^x = 0$ with initial value 0. | 4 | 2,3 | 6 |
| c) | Prove that $\frac{sin6\theta}{sin2\theta} = 16cos^4\theta - 16cos^2\theta + 3$. | 6 | 4 | 1 |
| d) | If $y = 2x\sqrt{1-x^2}$,prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-4)y_n = 0$. | 6 | 4 | 3 |
| Q.3. | Attempt all | | | |
| a) | Expand $2x^3 + 7x^2 + x - 1$ in powers of $x - 2$. | 4 | 3,5 | 3 |
| b) | Solve $5sinhx - coshx = 5$, and hence find $tanhx$. | 4 | 3,4 | 2 |
| c) | Reduce the following matrix A to normal form and find its rank. Where $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -3 & 1 \\ -3 & 1 & 1 \\ -6 & 2 & 2 \end{bmatrix}$ | 6 | 4 | 5 |
| d) | Find extreme values of $x^3 + xy^2 + 21x - 12x^2 - 2y^2$. | 6 | 5 | 4 |

P. T. O.

| | | | | |
|---|---|---|---|---|
| Q.4. | **Attempt all** | | | |
| a) | If $\alpha$ and $\beta$ are the roots of the equation $x^2 - 2\sqrt{3}x + 4 = 0$. Find $\alpha^3 + \beta^3$ and $\alpha^3 - \beta^3$. | 4 | 1,3 | 1 |
| b) | If $u = sin^{-1}(x^2 + y^2)$ then find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$ | 4 | 3,4 | 4 |
| c) | Solve the following system of equations by Gauss Seidel method $10x + 2y + z = 9, 2x + 20y - 2z = -44, -2x + 3y + 10z = 22$. | 6 | 3 | 6 |
| d) | By Principal of Mathematical induction prove that $1^2 + 2^2 + 3^2 + ---- + n^2 = \frac{n(n+1)(2n+1)}{6}$. | 6 | 5 | 2 |

CO1-Apply the basic concept of complex numbers and use it to solve problems in engineering.

CO2-Apply the basic concept of Hyperbolic, logarithmic functions and Logic in engineering problems.

CO3-Apply the concept of expansion of functions, successive differentiation and vector differentiation in optimization problems.

CO4-Use the basic concepts of partial differentiation in finding the Maxima and Minima required in engineering problems

CO5-Use the concept of matrices in solving the system of equations used in many areas of research...

CO6-Apply the concept of numerical Methods for solving the engineering problems with the help of SCILAB software.

BT Levels: - 1 Remembering ,2 Understanding, 3 Applying,4 Analyzing, 5 Evaluating, 6 Creating.

M-Marks, BT- Bloom's Taxonomy, CO-Course Outcomes.

PCE, New Panvel (Autonomous)

End Semester Examination 2nd Half 2021.

Branch:- F.Y.B.Tech (Comp/IT/ECS)

Subject:- Engineering Mathematics I    D.O.E. 04/04/22

Max. Marks:- 60    Time 11:00 to 1:00pm

Q.1 a) $x^5 - 1 = 0$

$\therefore x^5 = 1$

$\therefore x = (1)^{1/5} = (\cos 0 + i\sin 0)^{1/5} = (\cos 2\pi k + i\sin 2\pi k)^{1/5}$

$\therefore x = \cos\frac{2\pi k}{5} + i\sin\frac{2\pi k}{5}$.

put $k=0$, $x_0 = \cos 0 + i\sin 0 = 1$

$k=1$, $x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \alpha$

$k=2$, $x_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} = \alpha^2$

$k=3$, $x_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5} = \alpha^3$

$k=4$, $x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = \alpha^4$

$\because 1, \alpha, \alpha^2, \alpha^3$ & $\alpha^4$ are roots of $x^5 - 1 = 0$ — (3)

$\therefore x^5 - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$

$\therefore (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4) = \frac{x^5-1}{x-1}$

$\therefore (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4) = x^4 + x^3 + x^2 + x + 1$

put $x=1$

$\therefore (1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 1+1+1+1+1 = 5$ — (2)

Q.1b. $\bar{f} = x^2 z i - 2y^3 z^2 j + x y^2 z k$

$\therefore \operatorname{div} \bar{f} = \nabla \cdot \bar{f}$

$$= \frac{\partial \bar{f}_1}{\partial x} + \frac{\partial \bar{f}_2}{\partial y} + \frac{\partial \bar{f}_3}{\partial z}$$

$$= 2xz - 6y^2 z^2 + xy^2$$

$\therefore (\operatorname{div} \bar{f})_{(2,3,4)} = (2)(2)(4) - 6(9)(16) + (2)(9)$

$= 16 - 54 \times 16 + 18 = -830$ — 2

& $\operatorname{curl} \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 z & -2y^3 z^2 & x y^2 z \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(x y^2 z) - \frac{\partial}{\partial z}(-2y^3 z^2)\right]$$

$$-j\left[\frac{\partial}{\partial x}(x y^2 z) - \frac{\partial}{\partial z}(x^2 z)\right]$$

$$+k\left[\frac{\partial}{\partial x}(-2y^3 z^2) - \frac{\partial}{\partial y}(x^2 z)\right]$$

$$= i(2xyz + 4y^3 z) - j(y^2 z - x^2) + k(0)$$

$\therefore (\nabla \times \bar{f})_{(2,3,4)} = i(2(2)(3)(4) + 4(3)^3(4)) - j(9 \times 4 - 4) + k(0)$

$= i(48 + 27 \times 16) - j(36 - 4) + k(0)$

$= i(480) - j(32) + 0 \cdot k$

$= 480i - 32j + 0 \cdot k$ — 3

Q.1 c) $u = f(ax-by,\ by-cz,\ cz-ax)$

Let $p = ax-by$, $q = by-cz$, $r = cz-ax$

$\because u = f(p,q,r)$ & $p,q,r$ are functions of $x,y,z$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$$

$$= \frac{\partial u}{\partial p}(a) + \frac{\partial u}{\partial q}\cdot(0) + \frac{\partial u}{\partial r}(-a) \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}$$

$$= \frac{\partial u}{\partial p}(-b) + \frac{\partial u}{\partial q}\cdot(b) + \frac{\partial u}{\partial r}\cdot(0) \quad \text{--- (2)}$$

$$\& \ \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$= \frac{\partial u}{\partial p}\cdot(0) + \frac{\partial u}{\partial q}\cdot(-c) + \frac{\partial u}{\partial r}\cdot(c) \quad \text{--- (3)}$$

--- (3)

$\therefore$ from (1), (2) & (3)

$$\frac{1}{a}\frac{\partial u}{\partial x} + \frac{1}{b}\frac{\partial u}{\partial y} + \frac{1}{c}\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} - \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} - \frac{\partial u}{\partial q} + \frac{\partial u}{\partial r}$$

$$= 0$$

--- (2)

Q.1 d. The given system of equations can be written as $AX = B$

$$\begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & -3 \\ 4 & -2 & 6 \\ 15 & -3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ 21 \end{bmatrix}$$

$R_2 - 4R_1, \quad R_3 - 15R_1$

$$\begin{bmatrix} 1 & 1 & -3 \\ 0 & -6 & 18 \\ 0 & -18 & 54 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 12 \\ 36 \end{bmatrix}$$

$R_3 - 3R_2$

$$\begin{bmatrix} 1 & 1 & -3 \\ 0 & -6 & 18 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 12 \\ 0 \end{bmatrix}$$

$\therefore$ Rank of $A = 2 =$ Rank of $[A\ B]$

$\therefore$ system of equations consistent

& $r = 2$, $n = 3$ $\because r < n$ $\therefore$ system of equations has infinite number of solutions. — 3

$\therefore$ $x + y - 3z = -1$ — 1

$-6y + 18z = 12$

Let $z = t$

$\therefore -6y = 12 - 18t$

$y = -2 + 3t$

put in 1

$\therefore x + (-2+3t) - 3t = -1$

$\boxed{\therefore x = 1}$

— 2

$\therefore x = 1,\ y = -2+3t,\ z = t.$

Q.2 a) We know.

$$A = P + Q$$

where $P = \frac{1}{2}(A + A^t)$ & $Q = \frac{1}{2}(A - A^t)$

$$P = \frac{1}{2}\left\{\begin{bmatrix} 1 & 5 & 7 \\ -1 & -2 & -4 \\ 8 & 2 & 13 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 8 \\ 5 & -2 & 2 \\ 7 & -4 & 13 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 4 & 15 \\ 4 & -4 & -2 \\ 15 & -2 & 26 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 15/2 \\ 2 & -2 & -1 \\ 15/2 & -1 & 13 \end{bmatrix} \quad \text{— (2)}$$

$$\& \; Q = \frac{1}{2}\left\{\begin{bmatrix} 1 & 5 & 7 \\ -1 & -2 & -4 \\ 8 & 2 & 13 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 8 \\ 5 & -2 & 2 \\ 7 & -4 & 13 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 6 & -1 \\ -6 & 0 & -6 \\ 1 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & -1/2 \\ -3 & 0 & -3 \\ 1/2 & 3 & 0 \end{bmatrix} \quad \text{— (2)}$$

$$\therefore A = \begin{bmatrix} 1 & 2 & 15/2 \\ 2 & -2 & -1 \\ 15/2 & -1 & 13 \end{bmatrix} + \begin{bmatrix} 0 & 3 & -1/2 \\ -3 & 0 & -3 \\ 1/2 & 3 & 0 \end{bmatrix}$$

Q.2 b) $f(x) = \cos x - xe^x$, $x_0 = 0$

$$f'(x) = -\sin x - xe^x - e^x$$

We know, Newton Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{— (1)}$$

put $n=0$ in ①

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)}$$

$$= 1$$

$$x_2 = 1 - \frac{f(1)}{f'(1)} = 0.6531$$

$$x_3 = 0.6531 - \frac{f(0.6531)}{f'(0.6531)} = 0.5313$$

② 

$$x_4 = 0.5313 - \frac{f(0.5313)}{f'(0.5313)} = 0.5179$$

$$x_5 = 0.5179 - \frac{f(0.5179)}{f'(0.5179)} = 0.5178$$

$$\therefore x_6 = 0.5178 - \frac{f(0.5178)}{f'(0.5178)} = 0.5178.$$

②

Q.2 c. We know

$\cos 6\theta + i\sin 6\theta = (\cos\theta + i\sin\theta)^6$ By De-Moivre's Theorem

$$= \cos^6\theta + 6\cos^5\theta(i\sin\theta) + 15\cos^4\theta(i\sin\theta)^2 + 20\cos^3\theta(i\sin\theta)^3$$

$$+ 15\cos^2\theta(i\sin\theta)^4 + 6\cos\theta(i\sin\theta)^5 + (i\sin\theta)^6$$

$$= (\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta)$$

$$+ i(6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta)$$

Equating Imaginary part, we get

$$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta \quad \text{— (3)}$$

$$\therefore \frac{\sin 6\theta}{\sin 2\theta} = \frac{6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta}{2\sin\theta\cos\theta}$$

$$= 3\cos^4\theta - 10\cos^2\theta\sin^2\theta + 3\sin^4\theta$$

$$= 3\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + 3(1-\cos^2\theta)^2$$

$$= 3\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 3(1-2\cos^2\theta+\cos^4\theta)$$

$$= 16\cos^4\theta - 16\cos^2\theta + 3 \quad \text{— (3)}$$

Q.2 d.) $y = 2x\sqrt{1-x^2}$

$$\therefore y_1 = 2x\cdot\frac{1}{2\sqrt{1-x^2}}\cdot(-2x) + 2\sqrt{1-x^2}$$

$$\therefore y_1\cdot\sqrt{1-x^2} = -2x^2 + 2(1-x^2)$$

$$= 2 - 4x^2$$

$$\therefore y_1\cdot\frac{1}{2\sqrt{1-x^2}}\cdot(-2x) + y_2\sqrt{1-x^2} = -8x$$

$$\therefore y_2\cdot\sqrt{1-x^2} - \frac{x y_1}{\sqrt{1-x^2}} = -8x$$

$$\therefore y_2(1-x^2) - x y_1 = -8x\sqrt{1-x^2} = -4y$$

(3)

By Leibnitz theorem

$$(uv)_n = u_n v + n\,u_{n-1}v_1 + \frac{n(n-1)}{2!}u_{n-2}v_2 + \frac{n(n-1)(n-2)}{3!}u_{n-3}v_3 + \cdots \quad (1)$$

$$\therefore y_{n+2}(1-x^2) + n\,y_{n+1}(-2x) + \frac{n(n-1)}{2!}y_n(-2) - y_{n+1}x - n\,y_n = -4y_n$$

$$\therefore y_{n+2}(1-x^2) + x\,y_{n+1}(-2n-1) + y_n(-n(n-1) - n + 4) = 0 \quad (2)$$

$\therefore (1-x^2)y_{n+2} - xy_{n+1}\cdot(2n+1) - (n^2-4)y_n = 0.$ (8)

Q. 3 a) $f(x) = 2x^3+7x^2+x-1$, $a=2$, $f(2)=45$

$f'(x) = 6x^2+14x+1$ $\qquad f'(2) = 53$

$f''(x) = 12x+14$ $\qquad f''(2) = 38$

$f'''(x) = 12$ $\qquad f'''(2) = 12$

We know. Taylor's Series. — (2)

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a)$$

$$\therefore f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2}f''(2) + \frac{(x-2)^3}{6}\cdot f'''(2)$$

$$= 45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$$ — (2)

Q. 3 b) $5\sinh x - \cosh x = 5$

$$\therefore 5\left(\frac{e^x - e^{-x}}{2}\right) - \left(\frac{e^x + e^{-x}}{2}\right) = 5$$

$$\therefore 5e^x - 5e^{-x} - e^x - e^{-x} = 10$$

$$\therefore 4e^x - 6e^{-x} = 10$$

$$\therefore 2e^x - 3e^{-x} = 5$$

$$\therefore 2e^{2x} - 5e^x - 3 = 0$$

$$\therefore e^x = \frac{5 \pm \sqrt{25+4\times2\times3}}{2(2)} = \frac{5\pm7}{4}$$

$$\therefore e^x = \frac{12}{4} = 3 \qquad e^x = -\frac{2}{4} = -\frac{1}{2}$$ — (2)

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

When $e^x = 3$.

$$\therefore \tanh x = \frac{3 - 1/3}{3 + 1/3} = \frac{8/3}{10/3} = \frac{8}{10} = \frac{4}{5},$$

When $e^x = -1/2$

$$\tanh x = \frac{-1/2 + 2}{-1/2 - 2} = \frac{3/2}{-5/2} = -3/5. \quad \text{— ②}$$

Q.3c.) $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -3 & 1 \\ -3 & 1 & 1 \\ -6 & 2 & 2 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & -3 & 1 \\ 2 & 3 & 1 \\ -3 & 1 & 1 \\ -6 & 2 & 2 \end{bmatrix} \xrightarrow[R_4 + 6R_1]{R_2 - 2R_1,\ R_3 + 3R_1} \begin{bmatrix} 1 & -3 & 1 \\ 0 & 9 & -1 \\ 0 & -8 & 4 \\ 0 & -16 & 8 \end{bmatrix}$$

$$C_2 + 3C_1,\ C_3 - C_1 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & -1 \\ 0 & -8 & 4 \\ 0 & -16 & 8 \end{bmatrix} \quad \text{— ②}$$

$C_2 \leftrightarrow -C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 9 \\ 0 & -4 & -8 \\ 0 & -8 & -16 \end{bmatrix} \quad R_3 + 4R_2,\ R_4 + 8R_2 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 9 \\ 0 & 0 & 28 \\ 0 & 0 & 56 \end{bmatrix}$$

— ②

$C_3 - 9C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 28 \\ 0 & 0 & 56 \end{bmatrix} \xrightarrow{R_3/28} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 56 \end{bmatrix}$$

$$R_4 - 56R_3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

2

$\therefore$ Rank of $A = \rho(A) = 3$.

Q.3 d) $f(x,y) = x^3 + xy^2 + 21x - 12x^2 - 2y^2$

$f = x^3 + xy^2 + 21x - 12x^2 - 2y^2$

$f_x = 3x^2 + y^2 + 21 - 24x$ , $f_y = 2xy - 4y$

$f_{xx} = 6x - 24$ $\qquad$ $f_{yy} = 2x - 4$

1

$f_{xy} = 2y$

Now. $f_x = 0$,

$3x^2 + y^2 + 21 - 24x = 0$.

When $y=0$, $3x^2 + 21 - 24x = 0$

$\therefore x^2 - 8x + 7 = 0$

$\therefore (x-7)(x-1) = 0$

$\therefore x = 1$ , $x = 7$

When $x = 2$, $3(4) + y^2 + 21 - 48 = 0$

$\therefore y^2 + 33 - 48 = 0$

$y^2 = 15$

$y = \pm\sqrt{15}$

$f_y = 0$

$2xy - 4y = 0$

$2y(x-2) = 0$

$\therefore y = 0$, $x = 2$

$\therefore$ The stationary points are. $(1,0)$, $(7,0)$, $(2,\sqrt{15})$, $(2,-\sqrt{15})$

3 +

At $(2, \sqrt{15})$

$r = f_{xx} = 6x-24 = 12-24 = -12$

$s = f_{xy} = 2y = 2\sqrt{15}$

$t = f_{yy} = 2x-4 = 0$

$\therefore rt - s^2 = -0 - 4(15) = -60$

$\therefore$ we reject this pair.

At $(2, -\sqrt{15})$

$r = 6x-24 = -12$

$s = 2y = -2\sqrt{15}$

$t = 2x-4 = 0$

$\therefore rt - s^2 = 0 - 60$

$\therefore$ we reject this pair.

At $(1,0)$

$r = 6x-24 = 6-24 = -18$

$s = 2y = 0$

$t = 2x-4 = 2-4 = -2$

$\therefore rt - s^2 = 36 > 0$ & $r = -18 < 0$

$\therefore f$ has Maxima at $(1,0)$

$\therefore f(1,0) = 1^3 + 0 + 21 - 12 - 0$

$= 22 - 12 = 10$

At $(7,0)$

$r = 6x-24 = 42-24 = 18$

$s = 2y = 0$

$t = 2x-4 = 10$

$\therefore rt - s^2 = 180 > 0$ & $r = 18 > 0$

$\therefore f$ has minima at $(7,0)$

$\therefore f(7,0) = (7)^3 + 0 + 21(7) - 12(49) - 0$

$= 343 + 147 - 588$

$= -98$ ← 2

$\therefore f$ has maxima at $(1,0)$ and maximum value is 10

$f$ has minima at $(7,0)$ and minimum value is $-98$.

Q.4 a) $x^2 - 2\sqrt{3}x + 4 = 0$

$$\therefore x = \frac{2\sqrt{3} \pm \sqrt{4(3)-16}}{2}$$

$$= \frac{2\sqrt{3} \pm \sqrt{-4}}{2}$$

$$= \frac{2\sqrt{3} \pm 2i}{2}$$

$$= \sqrt{3} \pm i$$

$\therefore \alpha = \sqrt{3}+i, \quad \beta = \sqrt{3}-i$

$\alpha = \sqrt{3}+i = r(\cos\theta + i\sin\theta)$

$r = \sqrt{3+1} = 2, \quad \theta = \tan^{-1} 1/\sqrt{3} = \frac{\pi}{6}$

$\therefore \alpha = 2(\cos\pi/6 + i\sin\pi/6)$

$\beta = 2(\cos\pi/6 - i\sin\pi/6)$ — ②

$\therefore$ i) $\alpha^3 + \beta^3 = 2^3(\cos\pi/6 + i\sin\pi/6)^3 + 2^3(\cos\pi/6 + i\sin\pi/6)^3$

$= 8[\cos\pi/2 + i\sin\pi/2 + \cos\pi/2 - i\sin\pi/2]$

$= 8(2\cos\pi/2) = 8(2)(0) = 0$

ii) $\alpha^3 - \beta^3 = 2^3(\cos\pi/6 + i\sin\pi/6)^3 - 2^3(\cos\pi/6 - i\sin\pi/6)^3$

$= 8[\cos\pi/2 + i\sin\pi/2 - \cos\pi/2 + i\sin\pi/2)$

$= i8(2)\sin\pi/2$

$= 16i$ — ②

Q.4 b) $u = \sin^{-1}(x^2+y^2)$

$\therefore f(u) = \sin u = x^2+y^2$

put $x = xt,\ y = yt$

$\therefore f(u) = x^2t^2 + y^2t^2 = t^2(x^2+y^2) = t^2 f(u)$

$\therefore f(u)$ is homogeneous function of degree 2.

We know,

If $z = f(u)$ is homogeneous fun of degree n then — ②

$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1]$ where $g(u) = n\frac{f(u)}{f'(u)}$

Here $f(u) = \sin u$. $\quad \therefore g(u) = 2\frac{\sin u}{\cos u} = 2\tan u$, $g'(u) = 2\sec^2 u$

$\therefore x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\tan u[2\sec^2 u - 1]$ — ②

$= 2\tan u[2(1+\tan^2 u) - 1] = 2\tan u(1 + 2\tan^2 u)$

$= 2\tan u + 4\tan^3 u$

Gauss seidal Method.

Q.4c.

$10x+2y+z=9 \rightarrow x = \frac{9-2y-z}{10}$ —— (1)

$2x+20y-2z=-44 \rightarrow y = \frac{-44-2x+2z}{20}$ —— (2)

$-2x+3y+10z=22 \rightarrow z = \frac{22+2x-3y}{10}$ —— (3)

1st iteration:-

put $y=0, z=0$ in R.H.S of (1)

$\therefore x_1 = \frac{9}{10} = 0.9$

put $x=0.9, z=0$ in R.H.S of (2)

$\therefore y_1 = \frac{-44-2(0.9)+0}{20} = -2.29$

put $x=0.9, y=-2.29$ in R.H.S of (3)

$\therefore z_1 = \frac{22+2(0.9)-3(-2.29)}{10} = 3.0670$ —(2)

2nd iteration:-

put $y=-2.29, z=3.0670$ in R.H.S of (1)

$\therefore x_2 = \frac{9-2(-2.29)-3.0670}{10} = 1.0513$

put $x=1.0513, z=3.0670$ in R.H.S of (2)

$\therefore y_2 = \frac{-44-2(1.0513)+2(3.0670)}{20} = -1.9984$

put $x=1.0513, y=-1.9984$ in R.H.S of (3)

$\therefore z_2 = \frac{22+2(1.0513)-3(-1.9984)}{10} = 3.0098$ —(2)

3rd iteration:-

put $y=-1.9984, z=3.0098$ in R.H.S of (1)

$x_3 = \frac{9-2(-1.9984)-3.0098}{10} = 0.9987 \approx 1$

Put $x = 0.9987$, $z = 3.0098$ in R.H.S of (2)

$$y = \frac{-44 - 2(0.9987) + 2(3.0098)}{20} = -1.9989 \approx -2$$

Put $x = 0.9987$, $y = -1.9989$ in R.H.S of (3)

$$\therefore z = \frac{22 + 2(0.9987) - 3(-1.9989)}{10} = 2.9994 \approx 3$$

$\therefore$ $x = 1$, $y = -2$, $z = 3$ is the solution. — (2)

Q.4 d.)

1) Let $P(n): 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

2) Let $n = 1$, $\therefore$ L.H.S of $P(1) = 1^2 = 1$

R.H.S of $P(1) = \frac{1(1+1)(2+1)}{6} = \frac{(1)(2)(3)}{6} = 1$ — (2)

$\therefore$ L.H.S = R.H.S $\therefore P(1)$ is true.

3) Assume $P(k)$ is true i.e $P(n)$ is true for $n = k$.

$$\therefore 1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

4) We have to prove $P(k+1)$ is true i.e $P(n)$ is true for $n = k+1$

i.e $1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$ — (2)

L.H.S of $P(k+1) = 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{k+1}{6}\left(k(2k+1) + 6(k+1)\right) = \frac{k+1}{6}\left(2k^2 + 7k + 6\right) = \left(\frac{k+1}{6}\right)(2k+3)(k+2)$$

& R.H.S of $P(k+1) = \frac{(k+1)(k+2)(2k+2+1)}{6} = \left(\frac{k+1}{6}\right)(k+2)(2k+3)$

$\therefore$ L.H.S = R.H.S $\therefore P(k+1)$ is true. — (2)

$\therefore$ By principle of mathematical Induction it follows that $P(n)$ is true for all $n \geq 1$.